Journal of Mathematical Sciences, Vol. 200, No. 6, August, 2014

DOMINO TILINGS AND DETERMINANTS

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Consider an arbitrary simply connected figure F on the square grid and its dual graph (vertices correspond to cells, edges correspond to cells sharing a common side). We investigate the relationship between the determinant of the adjacency matrix of the graph and the domino tilings of the figure F. We prove that in the case where all the tilings can be split into pairs such that the numbers of vertical dominoes in each pair differ by one, then $\det A_F = 0$. *And in the case where all the tilings except one can be split into such pairs,* $\det A_F = (-1)^s$, *where s is half the area of the figure F. Bibliography: 6 titles.*

1. Introduction

Papers of Zeilberger ([5]), Chaiken, and others contain a special technique for solving problems of linear algebra, which allows us to interpret these problems in terms of graph theory. We are trying to extend this technique and prove the following theorem.

Let F be a simply connected bounded figure on the square grid consisting of unit squares, G_F be its dual graph, i.e., the graph whose vertices correspond to the cells of the figure and an edge joins two vertices if and only if the corresponding cells share a common side. Denote by A_F the adjacency matrix of the graph G_F. We say that two tilings form a good pair if the difference of the numbers of vertical dominoes in these tilings equals 2.

Theorem 1.1. *If the set of all tilings of a figure F can be split into good pairs, then* $\det A_F = 0$. *If the set of all tilings except one can be split into good pairs, then* $\det A_F = (-1)^s$ *where s is half the area of the figure.*

In Sec. 2, we describe how to interpret the determinant in terms of 1-factors and the pfaffian. In Sec. 3, we introduce the "sign of a figure" and show how to calculate it. In Sec. 4, we prove the main theorem. In Sec. 5, we calculate the determinant of the adjacency matrices of figures close to rectangles.

2. Determinants and 1-factors

Let F be a simply connected figure on the square grid consisting of $2s(F)$ unit squares. Let G_F be the dual graph of F, i.e., the vertices of G_F correspond to the cells of F and the edges correspond to the pairs of cells that share a common side. Observe that the graph G_F is bipartite, its partition is determined by the checkerboard coloring of the figure F. Denote by $A_F = (a_{ij})$ the adjacency matrix of the graph G_F; the matrix A_F is symmetric. We also consider more general symmetric matrices $\widetilde{A}_F$, which can be obtained from A_F by replacing 1's with arbitrary real numbers. For simplicity, we interpret the graph G_F and its subgraphs as directed graphs, treating each undirected edge as a pair of edges with opposite directions. The matrix elements a_{ij} are interpreted as the weights of the corresponding edges.

Recall that a 1-*factor* of the directed graph G_F is a subgraph that has the same set of vertices as G_F and is such that each its vertex has one ingoing and one outgoing edge.

Let F consist of n cells; then $\widetilde{A}_F$ is an $n \times n$ matrix and

$$\det \widetilde{A}_F = \sum_{\pi} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i,\pi(i)}, \tag{1}$$

where the sum is taken over the set of all permutations of $\{1, \dots, n\}$, and $\operatorname{sgn}(\pi)$ denotes the sign of a permutation. Each nonzero summand in formula (1) uniquely determines a 1-factor of the graph G_F, namely, the directed subgraph consisting of the edges $i \to \pi(i)$ (if $a_{i,\pi(i)} \neq 0$, then such an edge does exist). We denote the 1-factor determined by a permutation π by the same letter π.

If a permutation π is written as a product of cycles, we can calculate its sign by the formula

$$\operatorname{sgn}(\pi) = (-1)^{l_1 - 1}(-1)^{l_2 - 1} \dots (-1)^{l_m - 1},$$

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Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 421, 2014, pp. 5–18. Original article submitted December 9, 2013.

1072-3374/14/2006-0647

where m is the number of cycles in the permutation and l_i is the length of the ith cycle. Our graph is bipartite, therefore, every cycle has even length. Each cycle of even length regarded as a permutation is an odd permutation. Thus we can calculate the sign of a permutation π by the formula

$$\operatorname{sgn}(\pi) = (-1)^{\text{number of cycles in } \pi},$$

and the determinant can be calculated by the formula

$$\det \widetilde{A}_F = \sum_{\pi} (-1)^{\text{number of cycles in } \pi} W_\pi, \tag{2}$$

where the sum is taken over the set of all 1-factors of the graph G_F, and W_π denotes the weight of the 1-factor π, which equals, by definition, the product of the weights of all its edges. For the adjacency matrix A_F, each factor W_π equals 1.

Definition. *In what follows, we use the term* configuration *instead of* 1-factor*, the parity of the number of cycles in a configuration is called the* parity *of the configuration, and the expression* $(-1)^{\text{number of cycles in } \pi}$ *is called the* sign *of a configuration* π.

For brevity, a domino tiling of a figure will be called merely a tiling. *Each tiling of a figure F consists of $s(F)$ dominoes. Each tiling of a figure F determines a perfect matching of the graph G_F; the word "perfect" will be omitted in what follows.*

Definition. *Fix a figure F. Denote by c_k the number of tilings of F that contain exactly k vertical dominoes. We call* $f_F(x) = \sum\limits_{i=0}^{+\infty} c_k \cdot x^k$ the polynomial of the vertical statistics of tilings of the figure F.

We say that an edge in a configuration is rising *if it is vertical and directed upwards, and* falling *if it is vertical and directed downwards. Denote by u_k the number of configurations in the figure F that contain exactly k rising edges. We call*

$$g_F(x) = \sum_{k=0}^{+\infty} u_k \cdot x^k \tag{3}$$

the polynomial of the vertical statistics of configurations of the figure F.

Theorem 2.1. *Let F be an arbitrary figure on the square grid consisting of unit squares. Then*

(1) *the number of configurations in the graph G_F is equal to the squared number of tilings of the figure F;*

(2) $g_F(x^2) = f_F^2(x)$.

Proof. 1. Consider the checkerboard coloring of the figure, and split the edges of each configuration into two groups: the edges that start at black vertices and the edges that start at white vertices. The edges of each group determine a matching, which can be interpreted as a tiling. This map is bijective.

2. Due to the above bijection, we see that the coefficient of x^k in $f_F(x)^2$ is equal to the number of configurations containing exactly k vertical edges. Since the number of rising edges in each configuration equals half the number of vertical edges, the assertion follows. □

It follows from the first claim of the previous theorem and formula (2) that the parity of the determinant $\det A_F$ is equal to the parity of the number of tilings of the figure F.

Let us remind the definition of the pfaffian. For each pair of vertices of an undirected graph G we fix the order in which the vertices of this pair should be written. We may assume that the vertices of the graph are numbered and, therefore, the corresponding order is given for each pair of numbers. The order on the pairs of vertices allows us to split the matchings of the graph into two classes. Two matchings belong to the same class if one of them, regarded as a set of ordered pairs of vertices, can be transformed into the other one by an even permutation. We can mark the matchings of the first class by the plus sign, and the mathings of the other class by the minus sign. Consider the skew symmetric matrix A in which $a_{ij} = -a_{ji}$ if (i, j) is a correctly ordered pair of vertices connected by an edge, and $a_{ij} = 0$ otherwise. We say that an edge (i, j) corresponds to the matrix element a_{ij} if (i, j) is a correctly ordered pair. Define the pfaffian of the matrix A as

$$\operatorname{Pf} A = \sum_{\tau} \operatorname{sgn}(\tau) w(\tau),$$

where the sum is taken over the set of all perfect matchings of the graph G, and $w(\tau)$ is the product of the matrix elements corresponding to the edges of the matching. It is known that

$$\det A = (\operatorname{Pf} A)^2. \tag{4}$$

For every figure on the square grid and its graph G_F there exists an orientation of the pairs of neighboring vertices such that all matchings of the figure F have the same sign; this orientation is called a *Pfaffian* orientation of the graph. If A is the skew symmetric adjacency matrix of the graph G_F constructed from a Pfaffian orientation, then $\operatorname{Pf} A$ is equal to the number of matchings of the graph. But we are interested in another, *non-Pfaffian*, orientation on the pairs of vertices.

Definition. *Consider the checkerboard coloring of the vertices of the graph G_F. Consider the orientation on the set of pairs of neighboring vertices such that the first vertex in each pair is black. We call it the* checkerboard orientation. *Denote by $\widetilde{A}_F^{\#}$ the skew symmetric matrix constructed from this orientation.*

Looking at the checkerboard coloring of the figure, we can represent $\widetilde{A}_F$ as a block 2×2 matrix, with the top left block corresponding to the black cells, and the bottom-right block corresponding to the white cells. If we change the signs of all matrix elements in "white" rows and "black" columns, we obtain a skew symmetric matrix $\widetilde{A}_F^{\#}$. Thus

$$\det \widetilde{A}_F = (-1)^{s(F)} \det \widetilde{A}_F^{\#}.$$

It is more convenient for us to explain this formula via pfaffians.

Theorem 2.2. *If F is an arbitrary figure on the square grid with even area $2s(F)$, then*

$$(-1)^{s(F)} \det \widetilde{A}_F = \left(\operatorname{Pf} \widetilde{A}_F^{\#}\right)^2.$$

Proof. The expression $(\operatorname{Pf} \widetilde{A}_F^{\#})^2$ counts all pairs of matchings (τ_1, τ_2) taken with the sign $\operatorname{sgn}(\tau_1)\operatorname{sgn}(\tau_2)$. The product of the signs is equal to 1 if one matching can be transformed into the other one by an even permutation, and -1, if this permutation is odd. To calculate $\operatorname{sgn}(\tau_1)\operatorname{sgn}(\tau_2)$, let us draw both matchings in our graph. The obtained picture is a set of cycles, i.e., a configuration; denote it by π (we draw its cycles without orientation, but formally the orientation of the cycles can be set as in the proof of the first claim of Theorem 2.1). The map $(\tau_1, \tau_2) \mapsto \pi$ is a bijection. Define the orientation of edges by the checkerboard coloring rule, i.e., the first vertex of an edge is always black. Now construct a permutation that transforms τ_1 into τ_2. First, perform the counterclockwise shift in each cycle; the obtained permutation has the parity $(-1)^{\text{number of cycles in } \pi}$. Under this shift τ_1 becomes τ_2, but all (!) the edges of the matching τ_2 are written in the wrong order, due to the properties of the checkerboard orientation. We correct this by applying transpositions, the parity of the corresponding permutation being equal to $(-1)^{s(F)}$. As a result, we have

$$\operatorname{sgn}(\tau_1)\operatorname{sgn}(\tau_2) = (-1)^{s(F)+\text{number of cycles in } \pi}. \tag{5}$$

Thus

$$\left(\operatorname{Pf} \widetilde{A}_F^{\#}\right)^2 = (-1)^{s(F)} \sum_{\pi} (-1)^{\text{number of cycles in } \pi} W_\pi.$$

The sum in the right-hand side is equal to $\det \widetilde{A}_F$ due to (2). ☐

3. The sign of a simply connected figure on the square grid

Lemma 3.1. *Let P be a simply connected polygon on the square grid. Let a (respectively, b) be the number of integer points with even (respectively, odd) ordinate on the boundary of P. Let d be the number of integer points inside P. Then the sum of the lengths of the vertical sides of P is equal to $a - b + 2d + 2$ modulo 4.*

Proof. Induction on the area. If the dual graph contains a pendant vertex, then cut off the corresponding cell. Otherwise cut off a suitable corner cell. ☐

Remark. We allow polygons to be degenerate (i.e., consider also cycles on two vertices, with two parallel edges). It is easy to see that the lemma remains true in the degenerate case.

Theorem 3.2. *Let F be a simply connected polygon on the square grid consisting of an even number of cells. Then either for every configuration in the graph G_F the parity of the number of rising edges is equal to the parity of the number of cycles in it, or for every configuration these numbers have opposite parities.*

Proof. Consider an arbitrary configuration in G_F. It is obvious that the number of rising edges in it equals the number of falling edges; denote this number by v. Let the configuration consist of k cycles. Each cycle is a polygon. Since all the cycles have even length, and the figure is simply connected, the inner part of each cycle contains an even number of integer points. Therefore, applying Lemma 3.1 to each cycle, we can omit the term $2d$ in the left-hand side of the congruence. Now, if we sum up over the set of all cycles, we obtain

$$A - B + 2 \cdot k \underset{\text{mod } 4}{\equiv} \text{the total length of all vertical sides } = 2 \cdot v,$$

where A is the number of integer points with even ordinates and B is the number of integer points with odd ordinates on the boundaries of all cycles. Since the configuration covers all integer points of the figure, the difference $A - B$ is even and does not depend on the configuration. Put $A - B = 2 \cdot t$. Then $2 \cdot v \equiv 2 \cdot t + 2 \cdot k \pmod 4$, and so $v \equiv t + k \pmod 2$. Since t does not depend on the configuration, the theorem follows. □

Definition. *If the two parities in the statement of Theorem* 3.2 *coincide, we say that* the sign of the figure F *equals* 1, *otherwise the sign of the figure* F *equals* -1. *Thus, by definition, for each configuration* π *in the graph* G_F

$$(-1)^{\text{number of rising edges in } \pi} = \operatorname{sgn} F \cdot (-1)^{\text{number of cycles in } \pi}. \tag{6}$$

We will also consider the "logarithm" of the sign of F, *which we denote by* $\operatorname{Sign} F$. *By definition,* $\operatorname{Sign} F$ *is equal to* 0 *or* 1 *so that*

$$\operatorname{sgn} F = (-1)^{\operatorname{Sign} F}.$$

Lemma 3.3. *The sign of a simply connected figure* F *can be calculated as follows.*

1. $\operatorname{Sign} F = \frac{1}{2}(A - B)$, *where* A *is the number of integer points with even ordinates in the dual figure and* B *is the number of points with odd ordinates.*

2. $\operatorname{Sign} F$ *equals half the difference of the number of black vertices and the number of white vertices in the horizontal "zebra" coloring of* F.

3. $\operatorname{Sign} F$ *equals the parity of the number of horizontal dominoes in any tiling of* F.

Proof. 1. This follows from the proof of Theorem 3.2.

2. This is almost the same as Claim 1. The difference $A - B$ equals the difference of the numbers of black and white cells in the horizontal "zebra" coloring of F.

3. If we interpret the tiling as a configuration, the number of cycles in it equals the number of dominoes, and the number of rising edges is equal to the number of vertical dominoes. By definition, $\operatorname{sgn} F = -1$ if the parity of the number of rising edges in the configuration is not equal to the parity of the number of cycles, and $\operatorname{sgn} F = 1$ otherwise. Thus

$$\operatorname{Sign} F \underset{\text{mod } 2}{\equiv} \text{number of cycles} + \text{number of rising edges} = \text{number of dominoes} + \text{ number of vertical dominoes}.$$

It remains to observe that the vertical dominoes are counted in both summands, while the horizontal dominoes are counted only in the first one. □

In the definition of the pfaffian, we can arbitrarily attach the sign "plus" to one of the two classes of matchings, and the sign "minus" to the other one. In the case of the checkerboard orientation, we can attach these signs "geometrically." Denote by $V(\tau)$ and $H(\tau)$ the number of vertical and horizontal edges in a matching τ, respectively.

Lemma 3.4. *In the definition of the pfaffian, we can set the sign of a matching* τ *equal to* $(-1)^{\frac{1}{2}(H(\tau)+\operatorname{Sign} F)}$.

Proof. The sum $H(\tau) + \operatorname{Sign} F$ is even by Lemma 3.3. Consider two matchings τ_1, τ_2 and the configuration π determined by them as in the proof of Theorem 2.2. Recall that every matching contains $s(F)$ edges, the numbers $V(\tau_1)$ and $V(\tau_2)$ always have the same parity, and the number of rising edges in the configuration constructed from the two matchings equals $\frac{1}{2}(V(\tau_1) + V(\tau_2))$.

Let us check that the signs specified by the statement of the lemma agree with the parity of the permutation from the definition of the sign of a matching, i.e., the sum $s(F) +$ number of cycles π in Eq. (5) is even if and only if the signs are equal. This is true, because modulo 2 we have

$$
\begin{aligned}
\text{number of cycles in } \pi + s(F) &= \text{number of rising edges in } \pi + \operatorname{Sign} F + s(F) \\
&= \frac{1}{2}(V(\tau_1) + V(\tau_2)) + \operatorname{Sign} F + \frac{1}{2}(V(\tau_1) + H(\tau_1) + V(\tau_2) + H(\tau_2)) \\
&\equiv \frac{1}{2}(H(\tau_1) + \operatorname{Sign} F) + \frac{1}{2}(H(\tau_2) + \operatorname{Sign} F). \qquad \square
\end{aligned}
$$

3.1. Formulas for the determinant of the adjacency matrix. The following theorem reduces the calculation of $\det A_F$ to the investigation of the vertical statistics of the figure F.

Theorem 3.5. *For every simply connected figure F,*

$$\det A_F = \operatorname{sgn} F \cdot \sum_\pi (-1)^{\text{number of rising edges in } \pi}, \tag{7}$$

$$\det A_F = \operatorname{sgn} F \cdot g_F(-1) = \operatorname{sgn} F \cdot f_F^2(\mathbf{i}), \tag{8}$$

where g_F and f_F are the polynomials of the vertical statistics.

Proof. Comparing formulas (2) and (6), we automatically obtain (7). By formulas (7) and (3),

$$\det A_F = \operatorname{sgn} F \cdot \sum_\pi (-1)^{\text{number of rising edges in } \pi} = \sum_{i=0}^{+\infty} c_i \cdot (-1)^i = g_F(-1).$$

Substituting -1 instead of x^2 in g, we obtain $\det A_F = \operatorname{sgn} F \cdot g_F(-1) = \operatorname{sgn} F \cdot {f_F}^2(\mathbf{i})$. $\square$

Definition. *We say that a pair of tilings is good if the difference of the numbers of vertical dominoes in them is equal to* 2*.*

Theorem 3.6. *Let F be an arbitrary simply connected figure on the square grid consisting of $2s(F)$ squares. If the set of all tilings of F can be split into good pairs, then* $\det A_F = 0$*. If the set of all tilings except one can be split into good pairs, then* $\det A_F = (-1)^{s(F)}$*.*

Proof. Let us calculate $\det A_F$ by formula (8). If a good pair consists of a tiling with k vertical dominoes and a tiling with $k+2$ vertical dominoes, then its contribution to $f_F(\mathbf{i})$ is equal to $\mathbf{i}^k + \mathbf{i}^{k+2} = 0$. Therefore, all good pairs contribute zero to $f_F(\mathbf{i})$ and the first claim of the theorem follows.

If the set of all tilings except one can be split into good pairs, we denote the number of vertical and horizontal dominoes in the remaining tiling by v and h, respectively, $h + v = s(V)$. Then $f_F(\mathbf{i}) = \mathbf{i}^v$ by the previous argument, $\operatorname{sgn} F = (-1)^h$ by Lemma 3.3, and, therefore, $\det A_F = \operatorname{sgn} F \cdot {f_F}^2(\mathbf{i}) = (-1)^{h+v} = (-1)^{s(F)}$. $\square$

Thus, to calculate $\det A_F$, we must know whether the set of all tilings of the figure can be split into good pairs. The figure (not simply connected) for which the set of tilings cannot be split into good pairs is shown in Fig. 1.

It is clear from the proof that in terms of the vertical statistics, the set of tilings can be split into good pairs if and only if the polynomial $f_F(x)$ is divisible by $x^2 + 1$.

3.2. Application to "stamps" and rectangles

Definition. *An n-*stamp *is a figure that can be obtained from the $n \times n$ square by deleting some cells on its upper and right sides (so it looks like a postage stamp, but with irregular perforation along two of its sides). Let us number the rows of an n-stamp from bottom to top, and the columns from left to right. Each cell is determined by the numbers of its row and column. We say that an n-stamp is regular (Fig.* 4*) if it contains exactly one cell from the pair (n, i) and (i, n) if $i < n$, and does not contain the cell (n, n). Otherwise we say that the stamp is* irregular *(see Fig.* 2*).*

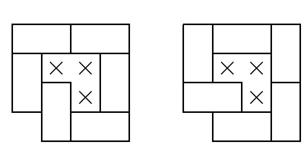

Fig. 1. The tilings of this figure cannot be split into good pairs.

Stamps were introduced by D. Karpov [2]; they are of interest because for them we know the parity of the number of tilings, namely, the following theorem holds.

Theorem 3.7 (see [2]). *The number of tilings of an n-stamp is odd if and only if the stamp is regular.*

Lemma 3.8 ("Half-diagonal lemma"). *Let a figure F contain three diagonal rows of cells shown in Fig. 3 and the cells marked by crosses do not belong to F. Then the set of all tilings of F that do not contain the domino marked by bold circles can be split into good pairs.*

We say that the statement of the lemma holds for the bottom-right direction. We will apply this lemma in other diagonal directions, too. This lemma, in a slightly different form, is proven in [3, Lemma 2]; there it was applied to prove Theorem 3.7. We apply a similar reasoning in the following theorem.

Theorem 3.9. 1. *Let F be an arbitrary regular n-stamp. Then* $\det A_F = (-1)^{n(n-1)/2}$.

2. *If F is an irregular n-stamp, then* $\det A_F = 0$.

Proof. 1. The expression $n(n-1)$ in the formula equals the area of any regular stamp. By Theorem 3.6, it suffices to check that the set of all tilings of every regular stamp except one can be split into good pairs. We will check this by induction on n. The base is trivial.

The induction step, $n \to n+1$. Consider a regular $(n+1)$-stamp. We will split the set of its tilings into good pairs. For this we take a look at the bottom-right and upper-left corner cells of the $(n+1)$-stamp. One of these cells lies inside the $n \times n$ square, let it be the upper-left cell. Apply the half-diagonal lemma in the bottom-right direction starting from this cell. Then the set of tilings that do not contain the domino marked in the left part of Fig. 4 can be split into good pairs. Let us look at the tilings that contain this domino. Apply the half-diagonal lemma in the upper-left direction starting from the cell to the left of the domino marked in the middle part of Fig. 4. By this lemma, the set of tilings that do not contain the marked domino in the upper-left corner can be split into good pairs. If we look at the remaining tilings, they contain this domino. Apply the half-diagonal lemma once again in the bottom-right direction from the cell below the domino, and so on. As a result of numerous applications of the half-diagonal lemma, we split the set of tilings into pairs except for the tilings containing all the dominoes on the left and bottom sides of our $(n+1)$-stamp (Fig. 4, right). By the induction hypothesis, there is a bijection between the remaining tilings and the tilings of a regular n-stamp. Therefore, all tilings except one can be split into good pairs.

2. By Theorem 3.6, it suffices to check that the tilings of every irregular stamp can be split into good pairs. We will check this by induction on n. The base is trivial.

The induction step, $n-1 \to n$. Consider an arbitrary n-stamp. We mark some cells of its $n \times n$ square as in Fig. 5.

Consider the following cases.

1) The figure does not contain the cells 1 and 4. Consider the diagonal from 5 to 6. By the half-diagonal lemma, the set of all tilings can be split into pairs (because the marked domino does not belong to the figure). Similarly, we obtain the same if the four cells 1, 2, 3, 4 do not belong to the stamp.

2) The stamp contains the cell 1, but not the cell 4 (or vice versa). Consider the first case, the second one being similar. Apply the half-diagonal lemma in the direction from 6 to 1. As in the proof of the previous claim, we split the set of all tilings into pairs, except for those tilings for which the position of dominoes on the leftmost column and bottom row is fixed as in the right part of Fig. 4. The set of exceptional tilings can be split into good pairs by the induction hypothesis.

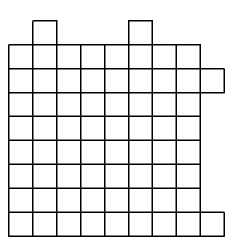

Fig. 2. An irregular 9-stamp.

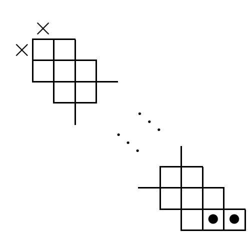

Fig. 3. The half-diagonal.

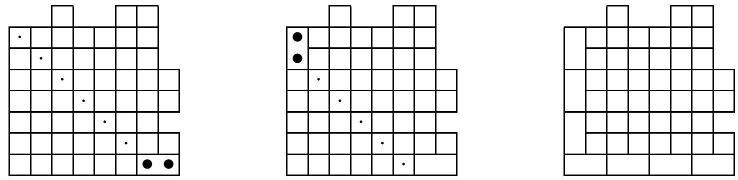

Fig. 4. Construction of an "unpaired" tiling of an $(n+1)$-stamp.

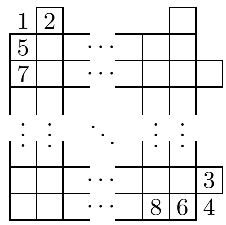

Fig. 5. The labels of the cells for an irregular stamp.

3) The cells 1 and 4 belong to the stamp, but the cells 2 and 3 do not. Then each tiling contains the dominoes 1–5 and 4–6. Cut them off. By the half-diagonal lemma, which we apply in the direction from 7 to 8, the set of all tilings can be split into good pairs.

4) The cells 1, 2, 4 belong to the stamp, but the cell 3 does not (or, similarly, 1, 3, 4 belong to the stamp, but 2 does not). Obviously, each tiling contains the domino 4–6. Cut it off. Apply the half-diagonal lemma in the direction from 8 to 7. Observe that each tiling contains the domino 5–7 and, therefore, each tiling contains the domino 1–2. We cut these dominoes off and complete the proof by induction, as in case 2. □

Note that this proof also proves Theorem 3.7. The following parity criterion for tilings of a rectangle is proven in [2].

Theorem 3.10. *The number of tilings of the $n \times m$ rectangle is odd if and only if the numbers $n+1$ and $m+1$ are coprime.*

In [3], this theorem is proven exactly by spliting tilings into good pairs! Combining Theorems 3.10 and 3.6, we obtain the following theorem.

Consequence 3.10.1. *For an arbitrary $m \times n$ rectangle,*

$$\det A_{n\times m} = \begin{cases} 0 & \text{if } \ (n+1, m+1) \neq 1; \\ (-1)^{\frac{n\cdot m}{2}} & \text{if } \ (n+1, m+1) = 1, \end{cases}$$

where (n, m) is the greatest common divisor of n and m.

This result is already known, see, for example, [6]. But the combinatorial proof presented here is new.

Translated by the authors.

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