REALIZATION OF SYSTEMS OF BOOLEAN FUNCTIONS BY LINEAR ARITHMETIC POLYNOMIALS

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This article proposes a canonical algorithmic structure that assures realization of Boolean functions in terms of linear arithmetic polynomials. Matrix relations for computation of the coefficients of the linear polynomials are derived. The authors present linearity conditions of different orders whose satisfaction permits simplification of the algorithmic structure. It is shown that the proposed approach is in some sense inherently more efficient than realization of systems of Boolean functions by threshold devices.

1. INTRODUCTION

The use of microprocessors and microcomputers in logic controls raises the possibility of using unconventional methods for realizing systems of Boolean function (SBF) by using the very general arithmetic capabilities of the indicated devices. One such method is realization of SBF as arithmetic polynomials (AP) [1-12].

In the general case, an AP is nonlinear and contains 2^n terms, so the computational complexity of AP grows rapidly as the number of variables increases.

It is therefore of particular interest to realize SBF by means of linear AP (LAP), since evaluation of an LAP reduces to summation of the coefficients of variables whose value is one.

Individual problems concerning realization of SBF as LAP were considered in [3, 7]. For example, [3] proposes an approach to realization of SBF by superposition of LAP that is based on the linear representability of certain classes of Boolean functions.

When this approach is used, the desired result is not constructed directly from LAP: it is constructed only after redundant bits in the binary representation of the value of the polynomial have been masked out.

A linearity condition (LC) whose satisfaction permits an SBF to be represented by a single LAP was found in [7]. However, only a limited class of SBF satisfies the LC.

In the present paper, we propose an approach that ensures, for systems not satisfying the LC, realization of arbitrary SBF by means of a set of LAP that is combined by means of a control structure (CS). The CS uses the value of an input set of values to choose one of the LAP from the set, which is then used to compute the decimal equivalent of the value of the SBF on the given input data.

2. ARITHMETIC POLYNOMIALS. FUNDAMENTALS

The notion of generalized modern disjunctive normal form (MDNF) for an SBF was introduced in [7]:

$$Y = y_0\bar{x}_1\bar{x}_2\ldots\bar{x}_n > y_1\bar{x}_1\bar{x}_2\ldots\bar{x}_{n-1}x_n \vee \ldots \vee y_{2^n-1}x_1x_2\ldots x_n, \quad (1)$$

"AVRORA" Scientific and Industrial Association of Central Scientific Research Institute, Saint Petersburg.
Translated from Avtomatika i Telemekhanika, No. 3, pp. 135-151, March, 1993. Original article submitted May 22, 1992.

Avtomation and Remote Control. Vol. 54, No3, Part2, March, 1993

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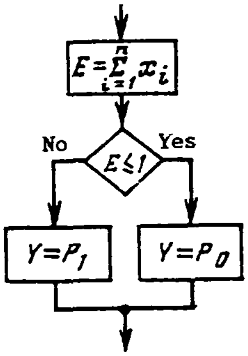

Fig. 1

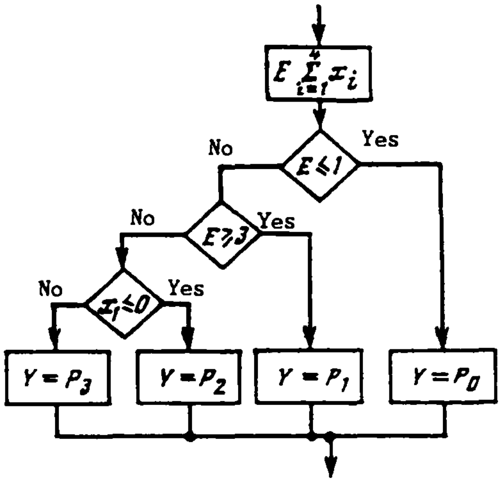

Fig. 2

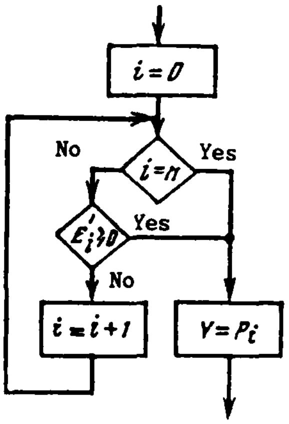

Fig. 3

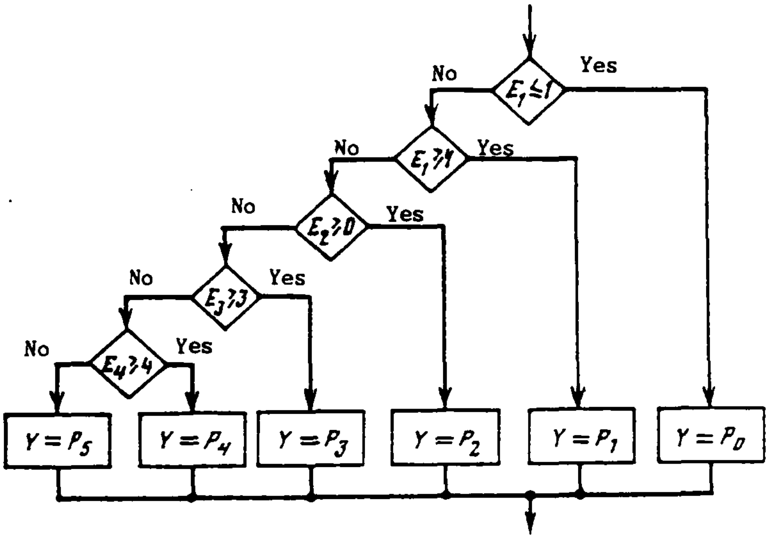

Fig. 4

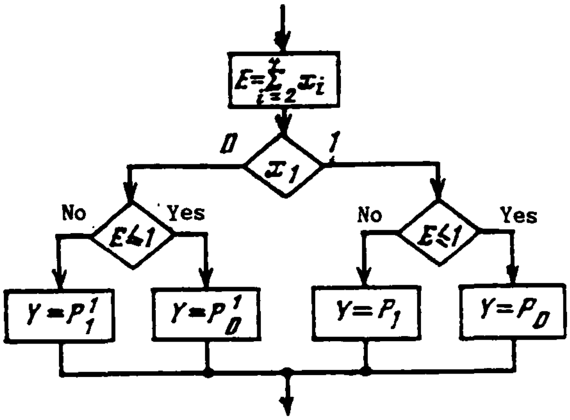

Fig. 5

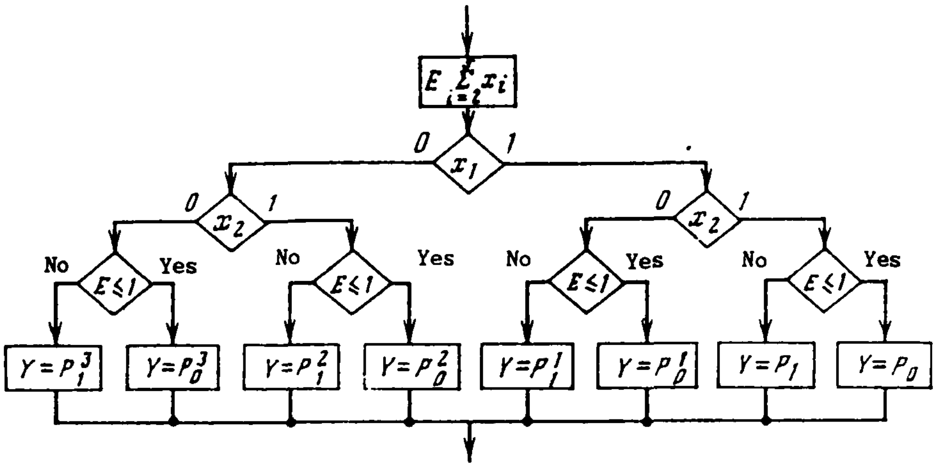

Fig. 6

where x_j is the j-th binary input variable ($j = 1,\ldots,n$); n is the number of input variables; y_i is the binary equivalent of the value of the SBF on the i-th set of input values for variables in a truth table (TT) $(i = \overline{0, 2^n - 1})$, $\bar{x}_j$ is the negation of the j-th input variable.

In the general case, an AP realizing an SBF is of the form

$$Y = a_0 + a_1 x_n + a_2 x_{n-1} + a_3 x_{n-1} x_n + \ldots + a_{2^n-1} x_1 x_2 \ldots x_n, \tag{2}$$

where a_i is some integer.

Expression (2) is constructed from (1) by substitution of a_i for the coefficients y_i and elimination of negated variables. Expressions (1) and (2) are related by matrix relations of the form

$$[A] = [K_n] \cdot [Y], \tag{3}$$

where $[A]$ and $[Y]$ are column vectors of the coefficients of an AP and MDNF, respectively.

For the matrix $[K_n]$ we have

$$[K_n] = \begin{bmatrix} K_{n-1} & 0 \\ -K_{n-1} & K_{n-1} \end{bmatrix}; \quad [K_1] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

where $[-K_{n-1}]$ is the matrix obtained by inverting the signs of the unit elements of the matrix $[K_{n-1}]$.

TABLE 1

Polynomial number	X	x_1 x_2 x_3	f_1 f_2 f_3	Y	Y_I	Y_{II}
0	0	0 0 0	0 0 0	0	y_0	–
0	1	0 0 1	0 1 1	3	y_1	–
0	2	0 1 0	0 1 1	3	y_2	–
1	3	0 1 1	0 1 0	2	–	y_3
0	4	1 0 0	0 1 1	3	y_4	–
1	5	1 0 1	0 1 0	2	–	y_5
1	6	1 1 0	0 1 0	2	–	y_6
1	7	1 1 1	1 1 1	7	–	y_7

TABLE 2

$Y =$

	1	x_3	x_2	$x_2 x_3$
1	0	3	3	–4
x_1	3	–4	–4	10

A software implementation of the indicated transformation was presented in [8]. We should note that the proposed approach makes it possible to realize SBF not only in terms of polynomials, but also in terms of tables with binary arguments and arbitrary real values of functions.

There is a matching inverse for transformation (3):

$$[Y] = [Q_n] \cdot [A],$$

where

$$[Q_n] = \begin{bmatrix} Q_{n-1} & 0 \\ Q_{n-1} & Q_{n-1} \end{bmatrix}; \quad Q_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

By an LAP we will mean an expression of the form

$$P = a_0 + a_1 x_n + a_2 x_{n-1} + \ldots + a_{n-1} x_1. \tag{4}$$

An LC for AP for SBF (a condition for representability of an SBF by expression (4)) was defined as follows in [7]:

$$[Y_l] = [DC] \cdot [Y_b], \tag{5}$$

where $[Y_l]$ is a column vector of $2^n - n - 1$ elements y_l $(l \neq 0, 2^0, 2^1, \ldots, 2^{n-1})$, in order of increasing indices; $[Y_b]$ is a column vector of $(n+1)$-th elements y_b $(b = 0, 2^0, 2^1, \ldots, 2^{n-1})$, in order of decreasing indices; D is a $(2^n - n - 1) \times (n + 1)$ matrix in which each row is the binary equivalent of l, which we denote by bin l; C is a column vector of $2^n - n - 1$ elements whose values are equal to 1 minus the sum of the ones in bin l.

For $n = 3$, expression (5) takes the form

$$\begin{bmatrix} y_3 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}. \tag{6}$$

TABLE 3

Polynomial number	X	$x_1\ x_2\ x_3\ x_4$	$f_1\ f_2$	Y
0	0	0 0 0 0	0 1	1
0	1	0 0 0 1	1 1	3
0	2	0 0 1 0	1 0	2
2	3	0 0 1 1	0 0	0
0	4	0 1 0 0	0 0	0
2	5	0 1 0 1	0 1	1
2	6	0 1 1 0	0 1	1
1	7	0 1 1 1	1 1	3
0	8	1 0 0 0	0 1	1
3	9	1 0 0 1	1 0	2
3	10	1 0 1 0	1 0	2
1	11	1 0 1 1	0 0	0
3	12	1 1 0 0	0 1	1
1	13	1 1 0 1	0 1	1
1	14	1 1 1 0	0 1	1
1	15	1 1 1 1	1 1	3

TABLE 4

$Y =$

	1	x_4	x_3	x_3x_4
1	1	2	1	−4
x_2	−1	−1	0	5
x_1	0	−1	0	1
x_1x_2	1	0	−1	0

TABLE 5

x_1x_2	$x_3x_4x_5$							
	000	001	011	101	110	111	101	100
00	–	–	1	–	1	1	1	–
01	–	0	1	0	0	–	0	0
11	0	0	–	0	–	–	–	0
10	–	0	0	0	0	–	0	0

TABLE 6

x_1x_2	$x_3x_4x_5$							
	000	001	011	010	110	111	101	100
00	–	–	–	–	–	–	–	–
01	–	0	–	0	0	–	0	0
11	0	0	–	1	–	–	–	0
10	–	1	1	1	1	–	0	0

By basic input sets (BIS) we will mean input sets in which the number of ones is no greater than one. The number of BIS for an SBF in n variables is $n + 1$. In this case the elements of the matrix $[Y_b]$ are the values of the SBF on the BIS.

If relation (5) is satisfied, then an SBF in n variables can be realized by a single LAP, which can be constructed from the values of the SBF on the BIS alone. We will call such an LAP basic. The coefficients of this polynomial are given by the relation

$$\begin{bmatrix} a_{2n-1} \\ a_{2n-2} \\ \vdots \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \ldots & 0 & 0 & -1 \\ 0 & 1 & 0 \ldots & 0 & 0 & -1 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 \ldots & 1 & 0 & -1 \\ 0 & 0 & 0 \ldots & 0 & 1 & -1 \\ 0 & 0 & 0 \ldots & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_{2n-1} \\ y_{2n-2} \\ \vdots \\ y_2 \\ y_1 \\ y_0 \end{bmatrix} \qquad (7)$$

which is obtained from (3) by eliminating all rows and columns of the matrix $[K_n]$ and the elements of the column vectors $[A]$ and $[Y]$ corresponding to nonlinear terms in the AP. The matrix used in (7) is $(n + 1) \times (n + 1)$.

3. REALIZATION OF SYSTEMS OF BOOLEAN FUNCTIONS OF TWO AND THREE VARIABLES

Suppose that we are given an SBF in two variables. Since, in this case, we have four input sets, of which three are BIS, the SBF can be realized in terms of two LAP of the form $P_0 = a_0 + a_1x_2 + a_2x_1$; $P_1 = y_3$. For $n = 2$, the coefficients of the polynomial P_0 are determined from (7) as follows:

$$\begin{bmatrix} a_2^0 \\ a_1^0 \\ a_0^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_2 \\ y_1 \\ y_0 \end{bmatrix}.$$

For example, an SBF taking the values $y_0 = 3$, $y_1 = 4$, $y_2 = 2$, and $y_3 = 5$ is realized by the polynomials $P_0 = 3 + x_2 - x_1$ and $P_1 = 5$. The LAP that are obtained are combined by a CS that defines the domain of actions of each. To construct it, we use the fact that each BIS contains no more than one one (Fig. 1 with $n = 2$).

We now turn to consideration of SBF in three variables. In order to do so, we represent the values of the SBF in the form of two incompletely defined functions, of which the first (Y_I) is formed from the values of the SBF on BIS, while the second (Y_{II}) is constructed from the remaining values (Table 1). As an example, this table also contains three Boolean function f_1, f_2, and f_3 that depend on the variables x_1, x_2, x_3.

The column Y_I is realized by an LAP whose coefficients are determined from (7) with $n = 3$:

$$\begin{bmatrix} a_4^0 \\ a_2^0 \\ a_1^0 \\ a_0^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}. \qquad (8)$$

Here the positions in the Y_I column in Table 1 that contain dashes correspond to values determined by the LAP. For the column Y_{II} we redefine the values of the SBF on a BIS so that the AP realizing this column is linear. To do this, we use a relation following from the LC (5):

$$[Y_b] = [DC]^{-1} \cdot [Y_I]. \qquad (9)$$

The authors have established that for $n = 3$, we have $[DC^{-1}] = [DC]$, so it follows from (6) that

$$\begin{bmatrix} y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} y_3 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} .$$

Substituting the relation we have found into (7) for $n = 3$, we obtain

$$\begin{bmatrix} a_4^1 \\ a_2^1 \\ a_1^1 \\ a_0^1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} y_3 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} . \tag{10}$$

Thus, the coefficients of an LAP with $n = 3$ are determined by (8) and (10). Since there is no difference between the number of LAP and the conditions used to choose them in the cases $n = 2$ and $n = 3$, we can use a flowchart (FC) (Fig. 1) to realize an SBF in three variables, provided we set $n = 3$.

Example 1. We must realize the SBF of Table 1. Using (8) and (10), we obtain $P_0 = 3x_3 + 3x_2 + 3x_1$; $P_1 = -8 + 5x_3 + 5x_2 + 5x_1$. To represent the LAP compactly, we introduce the matrix notation

$$\begin{bmatrix} P_0 \\ P_1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 & 3 \\ -8 & 5 & 5 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix} .$$

For the sake of comparison, we also realize the SBF in terms of a nonlinear AP (NAP), using (2) and (3) [7]:

$$Y = 3x_3 + 3x_2 - 4x_2x_3 + 3x_1 - 4x_1x_3 - 4x_1x_2 + 10x_1x_2x_3.$$

The authors propose a tabular notation (Table 2) as a compact notation for NAP.

It follows from our realization of an SBF in three variables in terms of NAP and LAP that the linear part of the NAP coincides with the basic LAP, and the nonlinear part is realized by means of a second LAP and a CS consisting of a third linear arithmetic polynomial and a conditional branch (CB).

4. CONSTRUCTION OF GROUPS OF VALUES FOR SYSTEMS OF BOOLEAN FUNCTIONS REALIZED BY ONE LINEAR POLYNOMIAL

We can always use (7) to realize the value of an SBF on a BIS with a single LAP. We now propose an approach to realization of an SBF by means of an LAP on the remaining input sets.

We divide these sets into the smallest number of groups such that each group contains no more than $(n + 1)$ input sets, and the values of the SBF in a group are realized by one LAP. This last condition is satisfied if, for the group of input sets under consideration, the matrix $[DC]$ is invertible, which makes it possible to use (9) and (7) to compute the LAP. Invertibility of the matrix $[DC]$ for a group containing $n + 1$ input sets is ensured by linear independence [13] of the system of equations defined by relation (5) on this group. The authors have established [14] that linear independence of the indicated system of equations is achieved when each group is constructed with the following relations:

$$X_{i+1} = X_i + F_i; \tag{11}$$

$$X_{i+1} = X_i - F_i, \tag{12}$$

where X_i is the index, in the TT, of the i-th input set in the group ($i = 1,2,\ldots,n + 1$); F_i is the Fibonacci number, which is given by the relations $F_i = F_{i-1} + F_j$, $F_1 = F_2 = 1$, $j = i - 1$ or $i - 2$. Here

$$F_i = 2F_{i-1}; \tag{13}$$

$$F_i = F_{i-1} + F_{i-2}. \tag{14}$$

Relation (11) corresponds to "passage" through the TT from top down, while (12) corresponds to "passage" from bottom up.

Example 2. Suppose that we are given an SBF in four variables that is realized on BIS by an AP. Of the remaining input sets, we must choose a group of (n + 1) elements on which the values of the SBF can be realized by another LAP.

Choosing the index of input sets with, for example, (11), we obtain $X_2 = X_1 + F_1$, $X_3 = X_2 + F_2$, $X_4 = X_3 + F_3$, $X_5 = X_4 + F_4$. Using relation (14), we find that $F_3 = 2$ and $F_4 = 3$. Then, for $X_1 = 5$, we obtain $X_2 = 6$, $X_3 = 7$, $X_4 = 9$, $X_5 = 12$.

We will verify invertibility of the matrix [DC] for the group of input sets that has been selected. To do this, we first write relation (5) for the group that has been obtained:

$$\begin{bmatrix} y_5 \\ y_6 \\ y_7 \\ y_9 \\ y_{12} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}.$$

Solving this system of equations, we obtain

$$\begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 2 & -2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_5 \\ y_6 \\ y_7 \\ y_9 \\ y_{12} \end{bmatrix}.$$

In our case, as a result, the inverse matrix exists, so we can use the last relation and (7) to construct an LAP for the group of input sets that has been found. In general, finding the inverse matrix is very difficult, but the authors have established [14] that with a group of (n + 1) input sets formed with (12) and (13), we have the following equation: $[DC]^{-1} = [DC]$.

5. REALIZING A SYSTEM OF BOOLEAN FUNCTIONS IN FOUR VARIABLES

In the Appendix we show that for $n = 4$, an arbitrary SBF can be realized by four LAP whose coefficients are computed with the following relations:

$$\begin{bmatrix} a_8^0 \\ a_4^0 \\ a_2^0 \\ a_1^0 \\ a_0^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}; \tag{15}$$

$$\begin{bmatrix} a_8^1 \\ a_4^1 \\ a_2^1 \\ a_1^1 \\ a_0^1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} y_7 \\ y_{11} \\ y_{13} \\ y_{14} \\ y_{15} \end{bmatrix}; \tag{16}$$

$$\begin{bmatrix} a_8^2 \\ a_4^2 \\ a_2^2 \\ a_1^2 \\ a_0^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_3 \\ y_5 \\ y_6 \end{bmatrix}; \tag{17}$$

$$\begin{bmatrix} a_8^3 \\ a_4^3 \\ a_2^3 \\ a_1^3 \\ a_0^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_9 \\ y_{10} \\ y_{12} \end{bmatrix}. \tag{18}$$

The LAP are combined by means of a CS (Fig. 2) that is based on the following features of the groups we have selected: The sets in the zeroth input set contain no more than one one; the sets in the first group contain no less than three ones; the sets in the second and third groups are separated by the value of the "leading" variable x_1.

Analysis of the FC (Fig. 2) shows that the reduction of all predicates to the form $E_i' \geq 0$ makes it possible to construct a cyclic FC (Fig. 3). For example, the predicate $E_0 = x_1 + x_2 + x_3 + x_4 \leq 1$ takes the form $E_0' \geq 0$ when $E_0' = 1 - x_1 - x_2 - x_3 - x_4$. In this case

$$\begin{bmatrix} E_0' \\ E_1' \\ E_2' \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \\ -3 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}.$$

Example 3. Find an LAP realizing the SBF specified by Table 3.

Using relations (15)-(18), we obtain

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & -1 \\ -4 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x_4 \\ x_3 \\ x_2 \end{bmatrix}.$$

For the sake of comparison, we write out an NAP for the SBF under consideration:

$$Y = 1 + 2x_4 + x_3 - 4x_3x_4 - x_2 - x_2x_4 + 5x_2x_3x_4 - x_1x_4 + x_1x_3x_4 + x_1x_2 - x_1x_2x_3.$$

This NAP can be written more compactly (Table 4).

It is clear from what we have said above that the proposed approach with a comparatively simple CS makes it possible to divide the values of the SBF into four groups, two of which are complete (contain $n + 1$ values), and two of which contain three values. Because no smaller number of groups can be found for $n = 4$ ($]\frac{16}{5}[= 4$), the authors attempted to divide the values of the SBF into groups so that three are complete, and the fourth contains a single value. To do so, we took the values of the SBF on the BIS (0, 1, 2, 4, 8), and for the remaining values we used a computer to construct all combinations of 11 items taken 5 a time ($C_{11}^5 = 462$) and for each of them, we tested linear independence of the equations in the LC. Generation of "independent" groups showed that there are many of them — 323.

To keep the CS relatively simple, among these groups we chose a set of five elements with three or more ones (7, 11, 13, 14, 15), which we used in an FC (Fig. 2), and for the remaining six values with two ones (3, 5, 6, 9, 10, 12) we constructed six groups of five elements ($C_6^5 = 6$), which were each tested for membership in the set of 323 "independent" groups. All of the groups proved to be "dependent," so it is not possible to solve the problem we have posed with a simple CS.

Analysis of the "independent" groups also established that among them there are disjoint elements such as (3, 6, 13, 14, 15), (5, 9, 10, 11, 12) and (5, 7, 11, 14, 15), (6, 9, 10, 12, 13). Each of these pairs (together with a basis and the group (7) in the first case, and the group (3) in the second) solves the problem stated, although with a very complex CS, which makes their use undesirable.

6. REALIZATION OF SYSTEMS OF BOOLEAN FUNCTIONS IN FIVE VARIABLES

The approach presented in §4 made it possible to divide all input sets of variables for $n = 5$ into six groups: (0, 1, 2, 4, 8, 16), (15, 23, 27, 29, 30, 31), (10, 18, 22, 24, 25, 26), (5, 13, 17, 19, 20, 21), (3, 7, 9, 11) and (6, 12, 14, 28), and for each of these a matrix relation was found for determination of the coefficients of an LAP for computation of the values of the SBF on input sets of the corresponding groups [14].

However, in contrast to the FC for realization of SBF in two, three, and four variables (Figs. 1 and 2) with our method for construction of groups, the CS for $n = 5$ contains not only LAP and CB, but other operators, which makes it impossible to recommend the use of LAP as desirable. We should note that the above approach for construction of groups of input sets causes an a priori unpredictability in the CS.

We will consider a different approach that is oriented toward obtaining CS consisting of only LAP and CB.

Here the zeroth group and the (first) group symmetric to it are found with the method given in §4. These groups are chosen by means of LAP of the form $E = \sum_{i=1}^{n} x_i$ and the predicates $E \leq 1$ and $E \geq n - 1$, respectively.

From (7) we obtain a relation of the following form for determining the coefficients of the LAP on the BIS:

$$\begin{bmatrix} a_{16}^0 \\ a_8^0 \\ a_4^0 \\ a_2^0 \\ a_1^0 \\ a_0^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_{16} \\ y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}. \tag{19}$$

For the first group, we use the approach considered in §4:

$$\begin{bmatrix} a_{16}^1 \\ a_8^1 \\ a_4^1 \\ a_2^1 \\ a_1^1 \\ a_0^1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} y_{15} \\ y_{23} \\ y_{27} \\ y_{29} \\ y_{30} \\ y_{31} \end{bmatrix}. \tag{20}$$

To form subsequent groups, we use a Carnot map [15], which defines a decision function for each group. Dashes are used to replace previously selected sets in the map. In the remaining positions of the map we attempt to locate no more than $(n + 1)$ ones corresponding to sets in the selected group (the remaining positions in the map are filled with zeros) so that the function will be a threshold function and assure linear independence of the system of equations defined by relation (5) on this group. The threshold property of the function makes it possible to realize it with a single LAP and one CB [16-18]. An efficient method for constructing threshold expressions for nonrepeating threshold formulas (NTF) is proposed in [14].

When a decision function satisfying the above conditions is found, the zeros are removed from the Carnot map, and the ones are replaced by dashes; this increases the indeterminacy in the decision function for the next group, which increases the probability that the function will be a threshold function. This procedure is repeated, and, in view of the increase in the indeterminacy, the chance of obtaining a threshold function (TF) increases in each step.

We divide the remaining twenty sets into four groups (with five in each). To determine a decision function for sets in the second group, we place dashes in the positions in the Carnot map corresponding to the sets in the zeroth and first groups. In the remaining groups we attempt to locate five ones and fifteen zeros so that the decision function for the sets in this group is a threshold function and the corresponding system of equations is linearly independent. The decision heuristic is shown in Table 5.

The function corresponding to this choice is of the form $f_2 = \bar{x}_1\bar{x}_2 \vee \bar{x}_1 x_4 x_5 = \bar{x}_1(\bar{x}_2 \vee x_4 x_5)$. Because this function is a threshold function realizing an SBF, it can be evaluated with the LAP $E_2 = -3x_1 - 2x_2 + x_4 + x_5$ and the predicate $E_2 \geq 0$. Thus, for the second group we use the sets 3, 5, 6, 7, and 11. For these sets (since $x_1 = 0$ and, consequently, $a_{16}^2 = 0$), (5) takes the form

$$\begin{bmatrix} y_3 \\ y_5 \\ y_6 \\ y_7 \\ y_{11} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} y_8 \\ y_4 \\ y_1 \\ y_1 \\ y_0 \end{bmatrix}.$$

Solving this system of equations, we obtain

$$\begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & -2 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_3 \\ y_5 \\ y_6 \\ y_7 \\ y_{11} \end{bmatrix}.$$

This relation, together with (7) and the fact that $a_{16}^2 = 0$, yields

$$\begin{bmatrix} a_{16}^2 \\ a_8^2 \\ a_4^2 \\ a_2^2 \\ a_1^2 \\ a_0^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_3 \\ y_5 \\ y_6 \\ y_7 \\ y_{11} \end{bmatrix}. \tag{21}$$

We now construct a Carnot map with dashes in place of the sets in the zeroth, first, and second groups, and we construct a decision function for the values of the third group (Table 6).

This function is realized by an NTF of the form $f_3 = x_1 x_4 \vee x_1\bar{x}_2\bar{x}_3 = x_1(x_4 \vee \bar{x}_2\bar{x}_3)$ and can be evaluated by means of the LAP $E_3 = 3x_1 - x_2 - x_3 + 2x_4$ and the predicate $E_3 \geq 3$. Thus, for the sets of the third group we use 17, 18, 19, 22, and 26.

Assuming that $y_0 = y_{16}$, and using transformations analogous to those used for the second group, we obtain

$$\begin{bmatrix} a_{16}^3 \\ a_8^3 \\ a_4^3 \\ a_2^3 \\ a_1^3 \\ a_0^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_{17} \\ y_{18} \\ y_{19} \\ y_{22} \\ y_{26} \end{bmatrix}. \tag{22}$$

TABLE 7

x_1x_2	$x_3x_4x_5$							
	000	001	011	010	110	111	101	100
00	–	–	–	–	–	–	–	–
01	–	1	–	0	1	–	1	1
11	0	1	–	–	–	–	–	0
10	–	–	–	–	–	–	0	0

We now construct a Carnot map with dashes in place of all the sets of the preceding groups. We indicate the sets of the fourth group by ones, and the members of the fifth by zeros (Table 7).

The function specified by this table is realized by an NTF of the form $f_4 = x_2x_5 \vee \bar{x}_1x_2x_3 = x_2(x_5 \vee \bar{x}_1x_3)$, which can be computed with the LAP $E_4 = -x_1 + 3x_2 + x_3 + 2x_5$ and the predicate $E_4 \geq 4$.

It follows from the above that in the fourth group we place the sets 9, 12, 13, 14, and 25, and in the fifth group we place 10, 20, 21, 24, and 28.

Assuming that $y_0 = 0$ and performing operations similar to those used for the second group, we obtain

$$\begin{bmatrix} a^4_{16} \\ a^4_8 \\ a^4_4 \\ a^4_2 \\ a^4_1 \\ a^4_0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_9 \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{25} \end{bmatrix}, \tag{23}$$

$$\begin{bmatrix} a^5_{16} \\ a^5_8 \\ a^5_4 \\ a^5_2 \\ a^5_1 \\ a^5_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_{10} \\ y_{20} \\ y_{21} \\ y_{24} \\ y_{28} \end{bmatrix}. \tag{24}$$

Thus, for the LAP in the CS we can write the matrix relation

$$\begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -2 & -3 \\ 0 & 2 & -1 & -1 & 3 \\ 2 & 0 & 1 & 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}. \tag{25}$$

The indicated LAP are used in an FC (Fig. 4).

We can thus assert that an arbitrary SBF in five variables can be realized by means of ten LAP and five predicates. The result we have obtained is important for the theory of TF, since it asserts that any SBF in five variables can be realized in a way that is in some sense simpler than ten "threshold" devices that each represent an LAP and predicate. Analysis of the FC (Fig. 4) shows that reduction of the predicates to the form $E_i' \geq 0$ also makes it possible to use a cyclic FC (Fig. 3 with $n = 5$). Here

$$\begin{bmatrix} E'_0 \\ E'_1 \\ E'_2 \\ E'_3 \\ E'_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 & -1 \\ -4 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -2 & -3 \\ -3 & 0 & 2 & -1 & -1 & 3 \\ -4 & 2 & 0 & 1 & 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}. \tag{26}$$

Example 4. We use the above relations to construct an LAP for an SBF for which the transpose of the column of values is of the form $[Y]^T$ = [2, 4, 4, 7, 5, 7, 7, 5, 4, 6, 6, 5, 7, 5, 5, 7, 4, 6, 6, 5, 7, 5, 5, 7, 6, 4, 4, 7, 5, 7, 7, 13].

In matrix form, the LAP obtained are written as follows:

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & 3 & 2 & 2 \\ -17 & 6 & 6 & 6 & 6 & 6 \\ 11 & -2 & -2 & -2 & -2 & 0 \\ 7 & -1 & -1 & -1 & -2 & 0 \\ 0 & -2 & -2 & -1 & 8 & -2 \\ 0 & -2 & 8 & -1 & -2 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix}.$$

Together with (25), these LAP are used in the FC of Fig. 4, while together with (26), they are used in the FC of Fig. 3.

7. REALIZATION OF SYSTEMS OF BOOLEAN FUNCTIONS OF n VARIABLES

For $n \geq 6$, finding relations for realization of SBF with a minimal number of LAP is very difficult. As a result, for $n \geq 6$, we propose use of the Shannon factorization in order to represent a given SBF in the form of a set of residual SBF that each depend on no more than five variables. In this case, construction of the LAP for realization of each of the SBF obtained is accomplished with the relations given in the preceding sections.

Of course, the Shannon factorization can also be used when $n < 6$. Figure 5 shows FC realizing arbitrary SBF for $n = 4$ in which the residuals are SBF in three variables. Comparison of the acyclic FC shown in Figs. 2 and 5 shows that they have the same complexity, but the latter is more homogeneous. Use of Shannon factorization with residual SBF in three variables for $n = 5$ leads to construction of an FC with the same number of LAP — nine — and seven CB (Fig. 6), while the FC of Fig. 4 contains nine LAP and five CB. It should be noted that the first form of FC contains a substantially smaller number of coefficients (35 instead of 53).

It follows from what we have said that the use of LAP for $n = 4$ and $n = 5$ in an acyclic FC has no particular advantage over structures using Shannon factorization and LAP for $n = 3$.

However, as Fig. 3 shows, cyclic realization yields a very efficient application of the relations found above for computing the coefficients of LAP for $n = 4$ and $n = 5$.

APPENDIX

CONSTRUCTION OF LAP FOR SBF IN FOUR VARIABLES

The zeroth group consists of the values of the SBF on a BIS. The coefficients of the LAP for this group are computed with (15).

The first group is constructed with (12) and (13). Setting $X_1 = 15$, we find that $X_2 = 14$, $X_3 = 13$, $X_4 = 11$, $X_5 = 7$. For these sets (5) takes the form

$$\begin{bmatrix} y_7 \\ y_{11} \\ y_{13} \\ y_{14} \\ y_{15} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 & -2 \\ 1 & 1 & 0 & 1 & -2 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & 1 & 1 & 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}.$$

Because $[DC]^{-1} = [DC]$,

$$\begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 & -2 \\ 1 & 1 & 0 & 1 & -2 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & 1 & 1 & 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} y_7 \\ y_{11} \\ y_{13} \\ y_{14} \\ y_{15} \end{bmatrix}. \tag{A.1}$$

Substituting the equation found into (7) with $n = 4$, we obtain (16).

We divide the remaining six sets into two groups, and determine LAP for each of them. Because the leading digit of sets 3, 5, and 6 is zero, and in sets 9, 10, and 12 it is one, we can form the second and third groups as follows: (3, 5, 6) and (9, 10, 12). We now write the LC for each of these groups:

$$\begin{bmatrix} y_3 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}, \tag{A.2}$$

$$\begin{bmatrix} y_9 \\ y_{10} \\ y_{12} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}. \tag{A.3}$$

Since the number of variables in the systems under discussion exceeds the number of equations, these systems may have multiple solutions. As one of them (for (A.2) and (A.3), respectively) we obtain

$$\begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_3 \\ y_5 \\ y_6 \end{bmatrix},$$

$$\begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_9 \\ y_{10} \\ y_{12} \end{bmatrix}.$$

Substituting these expressions into (7) with $n = 4$, we obtain (17) and (18).

USE OF LINEARITY CONDITIONS FOR SIMPLIFICATION OF REALIZATIONS (IN THE EXAMPLE OF SBF IN FOUR VARIABLES)

The algorithmic structure (Fig. 2) proposed in the present paper is universal for the indicated number of variables. However, when this structure is used, it is desirable to verify LC (5) for $n = 4$:

$$\begin{bmatrix} y_3 \\ y_5 \\ y_6 \\ y_7 \\ y_9 \\ y_{10} \\ y_{11} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{15} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 & -2 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 1 & -2 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & 1 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}; \qquad \text{(A.4)}$$

when this condition is satisfied, an SBF can be realized with one LAP (P_0). If the LC is not tested and we immediately apply relations for computation of the coefficients of LAP, the FC will contain more than one polynomial.

For example, for the SBF $[Y]^T = [5, 7, 6, 8, 2, 4, 3, 5, 6, 8, 7, 9, 3, 5, 4, 6]$ the LC is satisfied, so $P_0 = 5 + 2x_4 + x_3 - 3x_2 + x_1$.

If the LC (A.4) is not satisfied (as occurs for the SBF of Table 3), it is desirable to apply less restrictive constraints (a second-order LC LC2) permitting realization of the SBF with an FC (Fig. 1) containing two LAP, P_0 and P_1.

We will state an LC2 for $n = 4$. For this purpose, we write the LC (A.4) for the six input sets that are not contained in the zeroth and first groups (§5):

$$\begin{bmatrix} y_3 \\ y_5 \\ y_6 \\ y_9 \\ y_{10} \\ y_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_8 \\ y_4 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}. \qquad \text{(A.5)}$$

In order to find conditions under which the LAP realizing the values of the SBF on the sets of the first group will also assure satisfaction of the values of the SBF on the desired six sets, we substitute (A.1) into (A.5):

$$\begin{bmatrix} y_3 \\ y_5 \\ y_6 \\ y_9 \\ y_{10} \\ y_{12} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} y_7 \\ y_{11} \\ y_{13} \\ y_{14} \\ y_{15} \end{bmatrix}. \qquad \text{(A.6)}$$

This relation is the desired LC2.

For example, for an SBF of the form $[Y]^T = [3, 5, 4, 11, 0, 10, 11, 12, 4, 6, 7, 8, 6, 7, 8, 9]$, the LC is not satisfied, but the LC2 is.

Here $P_0 = 3 + 2x_4 + x_3 - 3x_2 + x_1$, $P_1 = 8 + x_4 + 2x_3 + x_2 - 3x_1$.

When the LC2 is not satisfied, we must test a third-order LC (LC3) that will assure realization of an SBF with three LAP. We will state such a condition.

For it to be satisfied, it is necessary for the values of the SBF on the sets of the third group to be realized by the LAP found for the second group. In §5, the second group was constructed of the sets with indices 3, 5, and 6, so, substituting (A.6) into relation (A.4), we obtain the LC3:

$$\begin{bmatrix} y_9 \\ y_{10} \\ y_{12} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y_3 \\ y_5 \\ y_6 \end{bmatrix} . \qquad (A.7)$$

For example, for an SBF of the form $[Y]^T = [3, 4, 5, 5, 4, 5, 6, 13, 4, 4, 5, 12, 5, 13, 6, 14]$, the LC and LC2 are not satisfied, but the LC3 is. Here $P_0 = 3 + x_4 + 2x_3 + x_2 + x_1$, $P_1 = 2 + 8x_4 + x_3 + 2x_2 + x_1$; $P_2 = 4 + x_3 + x_2$. In the FC of Fig. 2 we can eliminate the CB $x_1 \leq 0$ and the fourth polynomial.

We should note that when a different method is used to construct the groups, different second- and third-order LC's may be obtained.

Thus, we can assert that after satisfaction of a j-th order LC, we can eliminate, from the canonical algorithmic structure of Fig. 2, the LAP (and corresponding CB) with indices beginning with j.

Similar LC, including a fourth-order LC, can be obtained for $n = 5$.

REFERENCES

1. V. D. Malyugin, "Realization of Boolean functions with arithmetic polynomials," *Avtom. Telemekh.*, No. 4, 84-93 (1982).
2. V. D. Malyugin, "On polynomial realization of corteges of Boolean functions," *Dokl. Akad. Nauk SSSR*, **265**, No. 6, 1338-1341 (1982).
3. V. D. Malyugin, "Realization of corteges of Boolean functions by means of linear arithmetic polynomials," *Avtom. Telemekh.*, No. 2, 114-122 (1984).
4. V. D. Efremov, A. A. Kuz'min, and V. A. Stepanov, "Computation of logical functions with Rademacher transforms," *Avtom. Telemekh.*, No. 2, 105-113 (1984).
5. V. D. Malyugin, G. A. Kukharev, and V. P. Shmerko, "Transformation of polynomial forms of Boolean functions," *Inst. Control Science Preprint*, Moscow (1986).
6. V. D. Malyugin, "Realization of systems of logical functions by means of arithmetic polynomials," *Computers and Artificial Inteligence*, No. 6, 541-552 (1987).
7. V. L. Artyukhov, V. N. Knodrat'ev, and A. A. Shalyto, "Realization of Boolean functions with arithmetic polynomials," *Avtom. Telemekh.*, No. 4, 138-147 (1988).
8. V. A. Osipov, A. A. Shalyto, and V. N. Kodrat'ev, "Software implementation of logical control algorithms in marine systems: Methodological developments," *Inst. Povysh. Kvalifik. Rukovod. Rabot. Spetsial. Sudostroit. Promysh.*, Leningrad (1988).
9. A. A. Shalyto, "Realization of algorithms for marine control systems using microprocessor technology," *Inst. Povysh. Kvalifik. Rukovod. Rabot. Spetsial. Sudostroit. Promysh.*, Leningrad (1988).
10. V. P. Shmerko, "Synthesis of arithmetic forms of Boolean functions with Fourier transforms," *Avtom. Telemekh.*, No. 5, 134-142 (1989).
11. V. L. Artyukhov, V. N. Knodrat'ev, and A. A. Shalyto, "The use of linear arithmetic polynomials for realization of logical control systems," in: *Proc. Eleventh All-Union Conference on Control Problems* [in Russian], Akad. Nauk SSSR, Moscow (1989), pp. 495-496.
12. V. N. Kondrat'ev and A. A. Shalyto, "Realization of an algorithm for control of submersible drilling rigs by linear arithmetic polynomials," in: *Proc. Seventh All-Union Scientific and Technical Congress on Problems of Complex Automation* [in Russian], VNTO im. A. N. Krylova, Leningrad (1989), p. 79.
13. L. I. Golovina, *Linear Algebra With Applications* [in Russian], Nauka, Moscow (1985).

14. A. A. Shalyto and V. N. Kondrat'ev, "Methods for software implementation of logical controls for marine microprocessor systems," *Inst. Povysh. Kvalifik. Rukovod. Rabot. Spets. Sudostroit. Promysh.*, Leningrad (1990).
15. D. A. Postelov, *Logical Methods for Analysis and Synthesis of System* [in Russian], Énergiya, Moscow (1974).
16. S. Muroga, I. Toda, and M. Kondo, "Majority decision function of up to six variables," *Mathematics of Computation*, **16**, No. 80, 132-150 (1962).
17. M. Dertouzos, *Threshold Logic* [Russian translation], Mir, Moscow (1967).
18. E. A. Butakov, *Methods for Synthesis of Relay Devices Using Threshold Devices* [in Russian], Énergiya, Moscow (1970).