BOUNDS ON THE REALIZATION COMPLEXITY OF BOOLEAN FORMULAS BY TREE CIRCUITS OF TUNABLE MODULES

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Upper and lower bounds are established on the circuit realization complexity of Boolean formulas in the basis of universal tunable modules.

Extensive literature is now available on design and application of tunable logic modules (TLM) [1-4]. There are, however, virtually no published results on the complexity of circuit realizations in the TLM basis.

The present article partly fills this gap for the case when the functioning algorithm of the device is represented by an h-letter Boolean formula defined in a basis with associative two-place operations and realized by a tree circuit consisting of universal TLMs in the same class of formulas with k or fewer letters [4].

1. STATEMENT OF THE PROBLEM

Consider an h-letter Boolean formula $f(z_1, z_2, \ldots, z_n)$ defined in a basis with associative two-place operations (e.g., $\{\&, \vee, -\}$, $\{\&, \vee, -, \oplus\}$).

Also given is a collection of modules M realizing all the subformulas φ in this basis of length not exceeding k letters. A universal TLM in this class of formulas with k or fewer letters may be used to represent these modules. In what follows, we refer to it as the k-universal module.

From the set of circuit realizations of the formula $f(z_1, z_2, \ldots, z_n)$ in the basis M, we isolate the subset of tree circuits.

Definition. A tree circuit is a single-output loopless structure in which every input variable and the output of every element are connected directly with at most one input of a single element in the structure.

Among the tree realizations, there is at least one with a minimal number of modules.

Let us estimate the number of modules L(h; k) from the set M required to construct a given realization.

We denote the inputs of the tree structure by x_i, where $i = 1, \ldots, h$. Then the original formula $f(z_1, z_2, \ldots, z_n)$ is transformed into a repetition-free formula of the form $f(x_1, x_2, \ldots, x_h)$.

On the other hand, we know [4] that a module is universal in some class of k-letter formulas only if its generating function is a combination of k-letter repetition-free formulas in the same basis.

Our problem thus reduces to decomposition of a repetition-free formula into repetition-free subformulas.

2. THE FINDINGS

Proceeding to discuss the findings, we should note that the TLM inputs may act both as information inputs and as tuning inputs. The TLM is linked with information sources and preceding modules in the logic structure by means of the information inputs. A TLM logic structures may be represented omitting the tuning inputs if each module is marked with the formula that is realized.

In what follows, we will only focus on information inputs, which we call module inputs. The module inputs to which input variables are applied are called activated; the remaining inputs are called free.

Proposition 1. The number of modules in a tree structure is given by

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$$L=\frac{h-1}{k_{av}-1}$$

where k_{av} is the average number of activated inputs in one module.

Proposition 2. The minimal number of modules in a tree structure is given by

$$L_{min}=\left]\frac{h-1}{k-1}\right[,$$

where]A[denotes rounding to the nearest integer not less than A.

Proposition 3. The minimal average number of activated inputs in a single module in a tree structure is given by

$$K_{av\,min}=\begin{cases}\frac{k}{2}+1, & \text{if at least one of the numbers L or K is even;}\\ \frac{k}{2}+1-\frac{1}{2L}, & \text{if both L and K are odd,}\end{cases}$$

and the maximal number of modules in an optimal tree structure is given by

$$L_{max}=\left]\frac{2(h-1)}{k}\right[.$$

Propositions 1-3 are proved in the Appendix.

Thus the number of modules in a minimal tree structure is bounded by

$$\left]\frac{h-1}{k-1}\right[\leqslant L\leqslant\left]\frac{2(h-1)}{k}\right[.$$

A method for realization of formulas in this TLM basis which attains the above bounds is presented in [5].

In conclusion note that our bounds can be extended to a larger class of modules, namely universal modules in the class of all Boolean functions. Since these modules, in particular, are universal in the class of formulas in the basis $\{\&, \vee, -, \oplus\}$, we have the following bounds on the realization of an arbitrary n-variable function defined by an h-letter formula in the same basis:

$$\left]\frac{n-1}{k-1}\right[\leqslant L\leqslant\left]\frac{2(h-1)}{k}\right[.$$

APPENDIX

Proof of Proposition 1

Without loss of generality, we can consider the case when the formulas f and φ are positive monotone.

Suppose that some method (e.g., enumeration) applied to a given formula has produced a minimal tree circuit consisting of L(h; k) k-universal modules. Let us analyze the resulting circuit in order to estimate the number of constituent modules.

In a tree structure there is at least one element with all the activated inputs connected only to sources of input variables. We assign the number 1 to this element and denote the number of its activated inputs by k_1. Clearly, $k_1 \leq k$.

Element 1 realizes some repetition-free formula φ_1 consisting of k_1 letters. Substituting φ_1 for the corresponding group of letters in the original formula $f(x_1,\ldots,x_h)$, we obtain a new repetition-free formula $f_1(\varphi_1, x^{(1)})$ consisting of $h-k_1+1$ letters.

In the remaining part of the structure there is an element (to which we assign the number 2) whose input variables belong to the set formed by the $h-k_1$ free inputs left after the extraction of the first element plus the first-element output. Suppose that element 2 realizes a repetition-free formula φ_2 consisting of k_2 letters. Substituting φ_2 for the corresponding group of letters in the formula $f_1(\varphi_1, x^{(1)})$, we obtain either a formula $f_2(\varphi_1, \varphi_2, x^{(2)})$, or a formula $f_2(\varphi_2, x^{(3)})$, both consisting of $h-k_1-k_2+1+1 = h-k_1-(k_2-1)+1 = h-(k_1+k_2)+2$ letters.

Continuing the structural analysis, we finally "traverse" the entire L-element structure and arrive at the single output. At this point we have

$$h-\sum_{i=1}^{L}k_i+L=1, \quad (A.1)$$

where k_i is the number of activated inputs of element i.

If the number of activated inputs k_i in (A.1) is expressed in terms of the module inputs k and the number of free outputs is expressed by $j_i - k_i = k - j_i$, (A.1) takes the form

$$h-Lk+\sum_{i=1}^{L}j_i+L=1. \quad (A.2)$$

The total number of free inputs in the circuit is denoted by

$$j=\sum_{i=1}^{L}j_i, \quad (A.3)$$

and we obtain for the total number of elements

$$L=\frac{h-1+j}{k-1}. \quad (A.4)$$

In order to express the number of elements as a function of fewer parameters, we introduce the notion of the average number of activated inputs.

Definition. The average number of activated inputs of the structure elements is defined by

$$k_{av}=\sum_{i=1}^{L}k_i \Big/ L. \quad (A.5)$$

The equality (A.1) thus takes the form

$$h-L(k_{av}-1)=1. \quad (A.6)$$

Hence,

$$L=\frac{h-1}{k_{av}-1}. \quad (A.7)$$

This expression establishes the dependence of the total number of elements in a structure on the number of letters h in the formula and the average number of activated inputs in the structure k_{av}.

Proof of Proposition 2

Consider a structure to which all the inputs of each element are activated. In this case k_{av} attains its maximum value and is equal to k.

The number of elements L in this structure realizing an h-letter formula is minimal. It is obtained from (A.7) by substituting k for k_{av}:

$$L_{min}=\frac{h-1}{k-1}. \quad (A.8)$$

Since a structure may only contain an integer number of elements, which should be sufficient to realize the formula, we have

$$L_{min}=\left]\frac{h-1}{k-1}\right[. \quad (A.9)$$

We thus have the bound

$$\left]\frac{h-1}{k-1}\right[\leq L. \quad (A.10)$$

Analyzing the Lower Bound. Because of rounding to the nearest larger integer, the number of modules remains minimal also for k_{av} slightly less than k. This fact is demonstrated in Table 1, which lists the values of k_{av} for which the following equality just holds:

TABLE 1. Values of $k_{av} = f(h, k)$

h	k							
	3	4	5	6	7	8	9	10
10	2.8	4	4	5,5	5.5	5,5	5.5	10
20	2,9	3,71	4,8	5.75	5.75	7.35	7.35	7,35
30	2,93	9,3	4.62	5.8	6.8	6.8	8.25	8,25
40	2,95	4	4.9	5,87	6.57	7.5	8.8	8,8
50	2.96	3.9	4.76	5.9	6.45	8	8	9,15
60	2,965	3,95	4,94	5,92	6.9	7,55	8,37	9,4

$$\frac{h-1}{k_{av}-1} = \left]\frac{h-1}{k-1}\right[. \tag{A.11}$$

Examination of Table 1 shows that in most cases the minimal value of elements is robust to quite large changes in k_{av}.

Proof of Proposition 3

Consider an arbitrary Boolean formula in the basis of associative two-place operations, $f(z_1, z_2, \ldots, z_n)$. Transform it into a repetition-free formula $f(x_1, x_2, \ldots, x_h)$.

Definition. A subformula is called separable if the value of f does not change when the subformula is enclosed in parentheses.

In the transformed formula, identify all the maximal separable nonintersecting subformulas ψ_i, consisting of $1 \leq h_i \leq k$ letters. Then the formula f takes the following form:

$$f(x_1, x_2, \ldots, x_h) = \psi_1 * \psi_2 * \ldots * \psi_i * \psi_{i+1} * \ldots * \psi_r.$$

Here * relates to the previously mentioned basis of associative two-place operations.

Regardless of the specific order of operations, the formula contains at least two subformulas ψ_i and ψ_{i+1} joined by the symbol * which are not separated by precedence-changing parentheses.

LEMMA 1. The realization of subformula $\psi_i * \psi_{i+1}$ requires two k-universal modules only if

$$h_i + h_{i+1} \geq k+1. \tag{A.12}$$

Lemma 1 leads to a number of corollaries.

COROLLARY 1. For a pair of k-universal modules (j, j+1) realizing the subformula $\psi_i * \psi_{i+1}$ for which (A.12) is true we have

$$k_j + k_{j+1} \geq k+2. \tag{A.13}$$

COROLLARY 2. The average number of activated inputs for the above pair of modules is given by

$$k_{av}^{j;j+1} = \frac{k_j + k_{j+1}}{2} \geq \frac{k}{2} + 1. \tag{A.14}$$

COROLLARY 3. A pair of k-universal modules realizing the subformula $\psi_i * \psi_{i+1}$ can always be chosen so that in a module whose activated inputs are all connected with the inputs, the number of activated inputs is given by

$$k_j \geq \begin{cases} \dfrac{k+1}{2} & \text{for odd k;} \\ \dfrac{k}{2} + 1 & \text{for even k.} \end{cases} \tag{A.15}$$

This corollary follows from the fact that if $h_i + h_{i+1} \geq k+1$, then at least one of the terms h_i or h_{i+1} is not less than $\left]\frac{k}{2}\right[= \frac{k+1}{2}$ for odd k and is not less than $\frac{k}{2} + 1$ for even k.

We now continue the proof of Proposition 3.

Realize the subformula $\psi_i * \psi_{i+1}$ using a circuit consisting of two k-universal modules k_j, k_{j+1}, denoting its output by z_1.

The residual formula then takes the form

$$f=\psi_1 * \psi_2 * \ldots * z_1 * \ldots * \psi_r.$$

This formula is transformed so that all the separable nonintersecting subformulas are of maximal length not exceeding k. This involves incorporating z_1 (if possible) in one of the subformulas ψ. Lemma 1 and its corollaries can be applied in performing this transformation of the residual formula.

Thus, in the residual formulas we again identify a pair of subformulas whose realization requires a pair of modules (j+2, j+3) with number of inputs given by (A.13), (A.14) (see Corollaries 1 and 2 of Lemma 1).

Therefore, for structures containing an even number of k-universal modules, we can propose a constructive realization procedure, such that for each pair of modules (A.14) is satisfied and the pairs are disjoint.

Therefore, for optimal tree structures with an even number of k-universal modules, we have the inequality

$$k_{av} \geqslant k/2+1. \tag{A.16}$$

Hence,

$$L_{even} \leqslant \left] \frac{2(k-1)}{k} \right[. \tag{A.17}$$

In order to show that this bound in general cannot be improved, it suffices to give an example of a minimal structure on which this bound is attained.

Example. Realize the formula $f=(x_1x_2 \vee x_3x_4)x_5 \vee x_6x_7$ using 3-universal modules.

The minimal realization of this formula requires four modules: $\varphi_1=x_1x_2$; $\varphi_2=\varphi_1 \vee x_3x_4$; $\varphi_3=\varphi_2 x_5$; $\varphi_4=\varphi_3 \vee x_6x_7$.

Thus,

$$k_{av}=\frac{2+3+2+3}{4}=2.5=\left]\frac{3}{2}\right[+1; \quad L=\left]\frac{2\cdot 6}{3}\right[=4.$$

For even L we have established that the number of modules in a minimal tree realization is bounded by

$$\left]\frac{h-1}{k-1}\right[\leqslant L \leqslant \left]\frac{2(h-1)}{k}\right[.$$

Now suppose that the number of modules L in the minimal tree structure realizing some Boolean function is odd.

By Corollary 3 of Lemma 1, for every k there is a module such that all the activated inputs [the number of these inputs is given by (A.15)] are connected with the input variables.

Introduce this module in the structure, and then delete it. The remaining fragment of the structure contains an even number of modules equal to L−1.

By (A.16), we have for this fragment

$$k_{av} \geqslant k/2+1.$$

If k is even, then for the entire structure we have

$$k_{av} \geqslant \frac{\frac{k}{2}+1+(L-1)\left(\frac{k}{2}+1\right)}{L}=\frac{k}{2}+1. \tag{A.18}$$

Thus,

$$L_{odd} \leqslant \left]\frac{2(h-1)}{k}\right[. \tag{A.19}$$

If k is odd, then for the entire structure

TABLE 2. Number of Letters h in Formulas Realizable by the Upper Bound

k	L										
	1	2	3	4	5	6	7	8	9	10	11
2	2	3	4	5	6	7	8	9	10	11	12
3	2	4	5	7	8	10	11	13	14	16	17
4	3	5	7	9	11	13	15	17	19	21	23
5	3	6	8	11	13	16	18	21	23	26	28
6	4	7	10	13	16	19	22	25	28	31	34
7	4	8	11	15	18	22	25	29	32	36	39
8	5	9	13	17	21	25	29	33	37	41	45
9	5	10	14	19	23	28	32	37	41	46	50
10	6	11	16	21	26	31	36	41	46	51	56
11	6	12	17	23	28	34	39	45	50	56	61

$$k_{av} \geq \frac{\frac{k+1}{2} + (L-1)\left(\frac{k}{2}+1\right)}{L} = \frac{k}{2} + 1 - \frac{1}{2L}. \qquad \text{(A.20)}$$

Thus, for a minimal tree structure with an odd number of k-universal modules and odd k, we have the inequality

$$k_{av.\,min} < k/2+1.$$

Let us find an upper bound on L in this case. Substituting (A.20) in (A.7), we obtain

$$L_{up} = \frac{2(h-1)}{k-1/L_{up}}.$$

Solving this equation for L_{up}, we obtain

$$L_{up} = \frac{2h-1}{k}. \qquad \text{(A.21)}$$

Seeing that L_{up} and k are odd, we have

$$L_{up} = \left]\frac{2(h-1)}{k}\right[. \qquad \text{(A.22)}$$

Thus,

$$L_{odd} \leq \left]\frac{2(h-1)}{k}\right[. \qquad \text{(A.23)}$$

Combining inequalities (A.17), (A.19), (A.23), we obtain for all L and k

$$L \leq \left]\frac{2(h-1)}{k}\right[. \qquad \text{(A.24)}$$

Analyzing the Upper Bound. Let us establish for what L, k and h the upper bound is attained and consider some examples.

We have seen that for even L and arbitrary k, and also for odd L and even k, we have

$$L_{up} = 2(n-1)/k.$$

For these L and k, the sought values of h are given by

$$h = \frac{Lk}{2} + 1, \qquad \text{(A.25)}$$

assuming that h, L, and k are integers.

For odd L and k,